

Fast $(1 + \epsilon)$ -approximation of the Löwner extremal matrices of high-dimensional symmetric matrices

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Abstract

Matrix data sets are common nowadays like in biomedical imaging where the Diffusion Tensor Magnetic Resonance Imaging (DT-MRI) modality produces data sets of 3D symmetric positive definite matrices anchored at voxel positions capturing the anisotropic diffusion properties of water molecules in biological tissues. The space of symmetric matrices can be partially ordered using the Löwner ordering, and computing extremal matrices dominating a given set of matrices is a basic primitive used in matrix-valued signal processing. In this letter, we design a fast and easy-to-implement iterative algorithm to approximate arbitrarily finely these extremal matrices. Finally, we discuss on extensions to matrix clustering.

keywords : Positive semi-definite matrices, Löwner ordering cone, extremal matrices, geometric covering problems, core-sets, clustering.

1 Introduction: Löwner extremal matrices and their applications

Let $M_d(\mathbb{R})$ denote the space of *square* $d \times d$ matrices with real-valued coefficients, and $\text{Sym}_d(\mathbb{R}) = \{S : S = S^\top\} \subset M_d(\mathbb{R})$ the matrix vector space¹ of *symmetric* matrices. A matrix $P \in M_d(\mathbb{R})$ is said *Symmetric Positive Definite* [1] (SPD, denoted by $P \succ 0$) iff. $\forall x \neq 0, x^\top P x > 0$ and only *Symmetric Positive Semi-Definite*² (SPSD, denoted by $P \succeq 0$) when we relax the strict inequality ($\forall x, x^\top P x \geq 0$). Let $\text{Sym}_d^+(\mathbb{R}) = \{X : X \succeq 0\} \subset \text{Sym}_d(\mathbb{R})$ denote the space of positive semi-definite matrices, and $\text{Sym}_d^{++}(\mathbb{R}) = \{X : X \succ 0\} \subset \text{Sym}_d^+(\mathbb{R})$ denote the space of positive definite matrices. A matrix $S \in \text{Sym}_d(\mathbb{R})$ is defined by $D = \frac{d(d+1)}{2}$ real coefficients, and so is a SPD or a SPSP matrix. Although $\text{Sym}_d(\mathbb{R})$ is a *vector space*, the SPSP matrix space does not have the vector space structure but is rather an abstract *pointed convex cone* with *apex* the zero matrix $0 \in \text{Sym}_d^+(\mathbb{R})$ since $\forall P_1, P_2 \in \text{Sym}_d^+(\mathbb{R}), \forall \lambda \geq 0, P_1 + \lambda P_2 \in \text{Sym}_d^+(\mathbb{R})$. Symmetric matrices can be *partially* ordered using the *Löwner ordering*:³ $P \succeq Q \Leftrightarrow P - Q \succeq 0$ and $P \succ Q \Leftrightarrow P - Q \succ 0$.

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¹Although addition preserves the symmetric property, beware that the product of two symmetric matrices may be not symmetric.

²Those definitions extend to Hermitian matrices $M_d(\mathbb{C})$.

³Also often written Loewner in the literature, e.g., see [2].

When $P \succeq Q$, matrix P is said to *dominate* matrix Q , or equivalently that matrix Q is dominated by matrix P . Note that the difference of two SPSP matrices may not be a SPSP matrix.⁴ A non-SPSP symmetric matrix S can be dominated by a SPSP matrix P when $P - S \succ 0$.⁵

The *supremum* operator is defined on n symmetric matrices S_1, \dots, S_n (not necessarily SPSPs) as follows:

Problem 1 (Löwner maximal matrices)

$$\bar{S} = \inf\{X \in \text{Sym}(\mathbb{R}) : \forall i \in [n], X \succeq S_i\}, \quad (1)$$

where $[n] = \{1, \dots, n\}$.

This matrix $\bar{S} = \max(S_1, \dots, S_n)$ is indeed the “smallest”, meaning the *tightest upper bound*, since by definition there does not exist another symmetric matrix X' dominating all the S_i 's and dominated by \bar{S} . Trivially, when there exists a matrix S_j that dominates all others of a set S_1, \dots, S_n , then the supremum of that set is matrix S_j . Similarly, we define the *minimal/infimum matrix* \underline{S} as the tightest lower bound. Since matrix inversion reverses the Löwner ordering ($A \succ B \Leftrightarrow B^{-1} \succ A^{-1}$), we link those extremal supremum/infimum matrices when considering sets of invertible symmetric matrices as follows: $\underline{S} = (\max(S_1^{-1}, \dots, S_n^{-1}))^{-1}$. Extremal matrices are *rotational invariant* $\max(O^\top S_1 O, \dots, O^\top S_n O) = O^\top \times \max(S_1, \dots, S_n) \times O$, where O is any orthogonal matrix ($OO^\top = O^\top O = I$). This property is important in DT-MRI processing that should be invariant to the chosen reference frame.

Computing Löwner extremal matrices are useful in many applications: For example, in matrix-valued imaging [3, 4] (morphological operations, filtering, denoising or image pyramid representations), in formal software verification [5], in statistical inference with domain constraints [6, 7], in structure tensor of computer vision [8] (Förstner-like operators), etc.

This letter is organized as follows: Section 2 explains how to transform the extremal matrix problem into an equivalent geometric minimum enclosing ball of balls. Section 3 presents a fast iterative approximation algorithm that scales well in high-dimensions. Section 4 concludes by hinting at further perspectives.

2 Equivalent geometric covering problems

We build on top of [9] to prove that solving the d -dimensional Löwner maximal matrix amounts to either find (1) the minimal covering Löwner matrix cone (wrt. set containment \subseteq) of a corresponding sets of D -dimensional cones (with $D = \frac{d(d+1)}{2}$), or (2) the minimal enclosing ball of a set of corresponding $(D - 1)$ -dimensional “matrix balls” that we cast into a geometric *vector* ball covering problem for amenable computations.

2.1 Minimal matrix/vector cone covering problems

Let $\mathcal{L} = \{X \in \text{Sym}^+(d) : X \succeq 0\}$ denote the *Löwner ordering cone*, and $\mathcal{L}(S_i)$ the reverted and translated *dominance cone* (termed the penumbra cone in [9]) with apex S_i embedded in

⁴For example, consider $P = \text{diag}(1, 2)$ and $Q = \text{diag}(2, 1)$ then $P - Q = \text{diag}(-1, 1)$ and $Q - P = \text{diag}(1, -1)$.

⁵ For example, $S = \text{diag}(-1, 1)$ is dominated by $P = \text{diag}(1, 1)$ (by taking the absolute values of the eigenvalues of S).

the space of symmetric matrices that represents all the symmetric matrices dominated by S_i : $\mathcal{L}(S_i) = \{S \in \text{Sym}_d(\mathbb{R}) : S_i \succeq S\} = S_i \ominus \mathcal{L}$, where \ominus denotes the *Minkowski set subtraction operator*: $\mathcal{A} \ominus \mathcal{B} = \{a - b : a \in \mathcal{A}, b \in \mathcal{B}\}$ (hence, $\mathcal{L}(0) = -\mathcal{L}$). A matrix S dominates S_1, \dots, S_n iff. $\forall i \in [n], \mathcal{L}(S_i) \subseteq \mathcal{L}(S)$. In plain words, S dominates a set of matrices iff. its associated dominance cone $\mathcal{L}(S)$ covers all the dominance cones $\mathcal{L}(S_i)$ for $i \in [n]$. The dominance cones are “abstract” cones defined in the $d \times d$ symmetric matrix space that can be “visualized” as equivalent *vector* cones in dimension $D = \frac{d(d+1)}{2}$ using *half-vectorization*: For a symmetric matrix S , we stack the elements of the lower-triangular matrix part of $S = [s_{i,j}]$ (with $s_{i,j} = s_{j,i}$): $\text{vech}(S) = [s_{1,1} \dots s_{d,1} \ s_{2,2} \dots s_{d,2} \dots s_{d,d}]^\top \in \mathbb{R}^{\frac{d(d+1)}{2}}$. Note that this is not the unique way to half-vectorize symmetric matrices but it is enough for *geometric containment* purposes. Later, we shall enforce that the ℓ_2 -norm of vectors $\text{vech}(S)$ matches the Fröbenius matrix norm $\|\cdot\|_F$.

Let \mathcal{L}_v denotes the vectorized matrix Löwner ordering cone: $\mathcal{L}_v = \{\text{vech}(P) : P \succ 0\}$, and $\mathcal{L}_v(S)$ denote the *vector dominance cone*: $\mathcal{L}_v(S) = \{\text{vech}(X) : X \in \mathcal{L}(S)\}$. Next, we further transform this minimum D -dimensional matrix/vector cone covering problems as equivalent *Minimum Enclosing Ball* (MEB) problems of $(D - 1)$ -dimensional matrix/vector balls.

2.2 Minimum enclosing ball of ball problems

A *basis* \mathcal{B} of a convex cone \mathcal{C} anchored at the origin 0 is a convex subset $\mathcal{B} \subseteq \mathcal{C}$ so that $\forall x \neq 0 \in \mathcal{C}$ there exists a *unique decomposition*: $x = \lambda b$ with $b \in \mathcal{B}$ and $\lambda > 0$. For example, $\text{Sym}_1^+(\mathbb{R}) = \{P \in \text{Sym}^+(\mathbb{R}) : \text{tr}(P) = 1\}$ is a basis of the Löwner cone $\mathcal{L} = \text{Sym}^+(\mathbb{R})$. Informally speaking, a basis of a cone can be interpreted as a *compact* cross-section of the cone. The Löwner cone \mathcal{L} is a smooth convex cone with its interior $\text{Int}(\mathcal{L})$ denoting the space of positive definite matrices $\text{Sym}^{++}(\mathbb{R})$ (full rank matrices), and its border $\partial\mathcal{L} = \mathcal{L} \setminus \text{Int}(\mathcal{L})$ the *rank-deficient* symmetric positive semi-definite matrices (with apex the zero matrix 0 of rank 0). A point x is an *extreme element* of a convex set S iff. $S \setminus \{x\}$ remains convex. It follows from Minkowski theorem that every compact convex set S in a finite-dimensional vector space can be reconstructed as convex combinations of its extreme points $\text{ext}(S) \subseteq \partial S$: That is, the compact convex set is the closed convex hull of its extreme points.

A face $\mathcal{F} \subset \mathcal{C}$ of a closed cone \mathcal{C} is a subcone such that $x + y \in \mathcal{F} \rightarrow x, y \in \mathcal{F}$. The 1-dimensional faces are the *extremal rays* of the cone. The basis of the Löwner ordering cone is [10] $\mathcal{B}(\mathcal{C}) = \text{CH}(vv^\top : v \in \mathbb{R}^d, \|v\|_2 = 1)$. Other rank-deficient or full rank matrices can be constructed by convex combinations of these rank-1 matrices, the extremal rays.

For any square matrix $X = [x_{i,j}]$, the *trace operator* is defined by $\text{tr}(X) = \sum_{i=1}^d x_{i,i}$, the sum of the diagonal elements of the matrix. The trace also amounts to the sum of the eigenvalues $\lambda_i(X)$ of matrix X : $\text{tr}(X) = \sum_{i=1}^d \lambda_i(X)$. The basis \mathcal{B}_i of a dominance cone $\mathcal{L}(S_i)$ is $\mathcal{B}_i = \{S_i - \text{tr}(S_i) \times \mathcal{B}(\mathcal{L})\}$. Note that all the basis of the dominance cones lie in the *subspace* H_0 of symmetric matrices with zero trace. Let $\langle X, Y \rangle_F = \text{tr}(X^\top Y)$ denote the *matrix inner product* and $\|M\|_F = \sqrt{\langle M, M \rangle_F} = \sqrt{\sum_{i,j} m_{i,j}^2}$ the matrix *Fröbenius norm*. Two matrices X and Y are orthogonal (or perpendicular) iff. $\langle X, Y \rangle_F = 0$. It can be checked that the identity matrix I is perpendicular to any zero-trace matrix X since $\langle X, I \rangle_F = \text{tr}(X) = 0$. The center of the ball basis of the dominance cone $\mathcal{L} = \mathcal{L}(S)$ is obtained as the *orthogonal projection* of S onto the zero-trace subspace H_0 : $\sigma(S) = S - \frac{\text{tr}(S)}{d}I$. The dominance cone basis is a *matrix ball* since for any rank-1 matrix $E = vv^\top$ with $\|v\|_2 = 1$ (an extreme point), we have the radius:

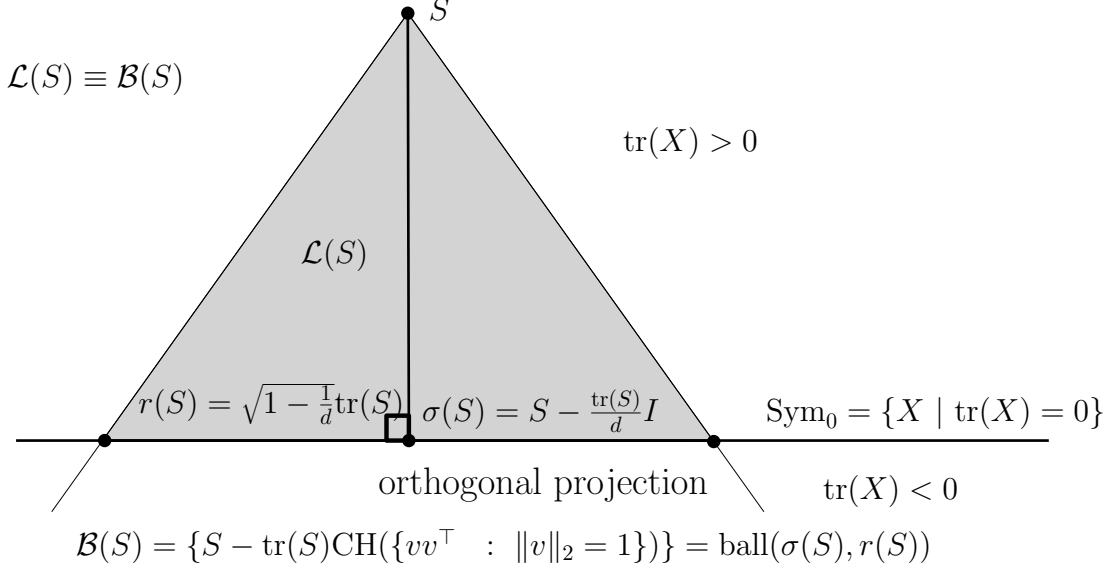


Figure 1: The dominance cone $\mathcal{L}(S)$ associated with matrix S has apex S and base $\mathcal{B}(S) = \text{Ball}(\sigma(S), r(S))$, a ball centered at matrix $\sigma(S)$ of radius $r(S)$. The cone $\mathcal{L}(S)$ has an equivalent representation $\mathcal{B}(S)$ provided that $\text{tr}(S) \geq 0$.

$$r(S) = \|S - \text{tr}(S)vv^\top - \sigma(S)\|_F = \text{tr}(S)\sqrt{1 - \frac{1}{d}}, \quad (2)$$

that is non-negative since we assumed that $\text{tr}(S) \geq 0$. Reciprocally, to a basis ball $B = \text{Ball}(\sigma, r)$, we can associate the apex of its corresponding dominance cone $\mathcal{L}(B)$: $\sigma + \frac{r}{d}\frac{I}{\sqrt{1 - \frac{1}{d}}}$. Figure 1 illustrates the notations and the representation of a cone by its corresponding basis and apex. Thus we associate to each dominance cone $\mathcal{L}(S_i)$ its corresponding ball basis $B_i = \text{Ball}(\sigma(S_i), r_i)$ on the subspace H_0 of zero trace matrices: $\sigma_i = \sigma(S_i) = S_i - \frac{\text{tr}(S_i)}{d}I$, $r_i = r(S_i) = \text{tr}(S_i)\sqrt{1 - \frac{1}{d}}$. We have the following containment relationships: $P \succ Q \Leftrightarrow \mathcal{L}(P) \supset \mathcal{L}(Q) \Leftrightarrow B(P) \supset B(Q)$ and $P \succeq Q \Leftrightarrow \mathcal{L}(P) \supseteq \mathcal{L}(Q) \Leftrightarrow B(P) \supseteq B(Q)$.

Finally, we transform this minimum enclosing *matrix* ball problem into a minimum enclosing *vector* ball problem using a half-vectorization that preserves the notion of distances, *i.e.*, using an isomorphism between the space of symmetric matrices and the space of half-vectorized matrices. The ℓ_2 -norm of the vectorized matrix should match the matrix Fröbenius norm: $\|s\|_2 = \|\text{vec}^+(S)\|_2 = \|S\|_F$. Since $\|S\|_F = \sqrt{\sum_{i=1}^d \sum_{j=1}^d s_{i,j}^2} = \sqrt{\sum_{i=1}^d s_{i,i}^2 + 2 \sum_{i=1}^{d-1} \sum_{j=i+1}^d s_{i,j}^2} = \|s\|_2$, it follows that $s = \|\text{vec}^+(S)\|_2 = [s_{1,1} \dots s_{d,d} \sqrt{2}s_{1,2} \sqrt{2}s_{1,3} \dots \sqrt{2}s_{d-1,d}]^\top \in \mathbb{R}^{\frac{d(d+1)}{2}}$. We can convert back a vector $v \in \mathbb{R}^D$ into a corresponding symmetric matrix.

Since we have considered all dominance cones with basis rooted on $H_0^+ : \text{tr}(X) \geq 0$ in order to compute the ball basis as orthogonal projections, we need to *pre-process* the symmetric matrices to ensure that property as follows: Let $t = \min\{\text{tr}(S_1), \dots, \text{tr}(S_n)\}$ denote the minimal trace of the input set of symmetric matrices S_1, \dots, S_n , and define $S'_i = S_i - tI$ for $i \in [n]$ where I denotes the identity matrix. Recall that $\text{tr}(X_1 + \lambda X_2) = \text{tr}(X_1) + \lambda \text{tr}(X_2)$. By construction, the transformed

input set satisfies $\text{tr}(S'_i) \geq 0, \forall i \in [n]$. Furthermore, observe that $S \succeq S_i$ iff. $S' \succeq S'_i$ where $S' = S - tI$, so that $\max(S_1, \dots, S_n) = \max(S'_1, \dots, S'_n) + tI$.

As a side note, let us point out that the reverse basis-sphere-to-cone mapping has been used to compute the convex hull of d -dimensional spheres (convex homothets) from the convex hull of $(d+1)$ -dimensional equivalent points [11, 12].

Finally, let us notice that there are several ways to majorize/minorize matrices: For example, one can seek extremal matrices that are invariant up to an *invertible transformation* [5], a stronger requirement than the invariance by orthogonal transformation. In the latter case, it amounts to geometrically compute the Minimum Volume Enclosing Ellipsoid of Ellipsoids (MVEEE) [5, 13].

2.3 Defining $(1 + \epsilon)$ -approximations of \bar{S}

First, let us summarize the algorithm for computing the Löwner maximal matrix of a set of n symmetric matrices S_1, \dots, S_n as follows:

1. Normalize matrices so that they have all non-negative traces:

$$S'_i = S_i - tI, \quad t = \min\{\text{tr}(S_1), \dots, \text{tr}(S_n)\}.$$

2. Compute the vector ball representations of the dominance cones:

$$B_i = \text{Ball}(\sigma_i, r_i)$$

with

$$\sigma_i = \text{vec}^+ \left(S'_i - \frac{\text{tr}(S'_i)}{d} I \right)$$

and

$$r_i = \text{tr}(S'_i) \sqrt{1 - \frac{1}{d}}$$

3. Compute the small(est) enclosing ball $B' = \text{Ball}(\sigma', r')$ of basis balls (either exactly or an approximation):

$$B' = \text{Small(est)EnclosingBall}(B_1, \dots, B_n)$$

4. Convert back the small(est) enclosing ball B' to the dominance cone, and recover its apex S' :

$$\bar{S}' = \sigma' + \frac{r'}{d} \frac{I}{\sqrt{1 - \frac{1}{d}}}.$$

5. Adjust back the matrix trace:

$$\bar{S} = \bar{S}' + tI, \quad t = \min\{\text{tr}(S_1), \dots, \text{tr}(S_n)\}.$$

Computing *exactly* the extremal Löwner matrices suffer from the *curse of dimensionality* of computing MEBs [14]. In [9], Burgeth et al. proceed by discretizing the basis spheres by sampling⁶

⁶In 2D, we sample $v = [\cos \theta, \sin \theta]^\top$ for $\theta \in [0, 2\pi[$. In 3D, we use spherical coordinates $v = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]^\top$ for $\theta \in [0, 2\pi[$ and $\phi \in [0, \pi[$.

the extreme x points vv^\top for $\|v\|_2 = 1$. This yields an approximation term, requires more computation, and even worse the method does not scale [15] in high-dimensions. Thus in order to handle high-dimensional matrices met in software formal verification [5] or in computer vision (structure tensor [8]), we consider $(1 + \epsilon)$ -approximation of the extremal Löwner matrices. The notion of tightness of approximation of \bar{S} (the epsilon) is imported straightforwardly from the definition of the tightness of the geometric covering problems. A $(1 + \epsilon)$ -approximation \tilde{S} of \bar{S} is a matrix $\tilde{S} \succ \bar{S}$ such that: $r(\tilde{S}) \leq (1 + \epsilon)r(\bar{S})$. It follows from Eq. 2 that a $(1 + \epsilon)$ -approximation satisfies $\text{tr}(\tilde{S}) \leq (1 + \epsilon)\text{tr}(\bar{S})$.

We present a fast guaranteed approximation algorithm for approximating the minimum enclosing ball of a set of balls (or more generally, for sets of compact geometric objects).

3 Approximating the minimum enclosing ball of objects and balls

We extend the incremental algorithm of Bădoiu and Clarkson [16] (BC) designed for finite point sets to *ball sets* or *compact object sets* that work in large dimensions. Let $B_1 = \text{Ball}(c_1, r_1), \dots, B_n = \text{Ball}(c_n, r_n)$ denote a set of n balls. For an object \mathcal{O} and a query point q , denote by $D^f(q, \mathcal{O})$ the *farthest* distance from q to \mathcal{O} : $D^f(q, \mathcal{O}) = \max_{o \in \mathcal{O}} \|q - o\|$, and let $F(q, \mathcal{O})$ denote the farthest point of \mathcal{O} from q . The generalized BC [16] algorithm for approximating the circumcenter of the minimum volume enclosing ball of n objects (MVBO) $\mathcal{O}_1, \dots, \mathcal{O}_n$ is summarized as follows:

- Let $e_1 \leftarrow x \in \mathcal{O}_1$ and $i \leftarrow 1$.
- Repeat l times:
 - Find the farthest object \mathcal{O}_f to current center: $f = \arg \max_{j \in [n]} D^f(e_i, \mathcal{O}_j)$
 - Update the circumcenter: $e_{i+1} = \frac{i}{i+1}e_i + \frac{1}{i+1}(F(e_i, \mathcal{O}_f) - e_i)$
 - $i \leftarrow i + 1$.

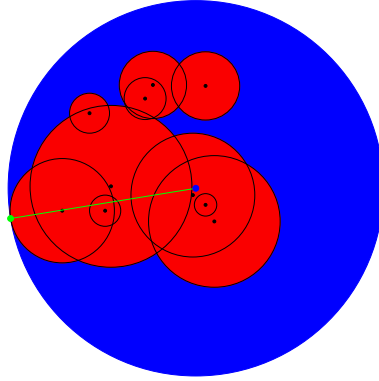
When considering balls as objects, the farthest distance of a point x to a ball $B_j = \text{Ball}(c_j, r_j)$ is $D^f(e_i, B_j) = \|c_j - e_i\| + r_j$, and the circumcenter updating rule is: $e_{i+1} = \frac{i}{i+1}e_i + \frac{1}{i+1}(c_f - e_i) \left(1 + \frac{r_f}{\|c_f - e_i\|}\right)$. See Figure 2 and online video⁷ for an illustration. (MVBO can also be used to approximate the MEB of ellipsoids.) It is proved in [17] that at iteration i , we have $\|e_i - e^*\| \leq \frac{r^*}{\sqrt{i}}$ where $B^* = \text{Ball}(e^*, r^*)$ is the unique smallest enclosing ball. Hence the radius of the ball centered at e_i is bounded by $(1 + \frac{1}{\sqrt{i}})r^*$. To get a $(1 + \epsilon)$ -approximation, we need $\frac{1}{\epsilon^2}$ iterations. It follows that a $(1 + \epsilon)$ -approximation of the smallest enclosing ball of n D -dimensional balls can be computed in $O(\frac{D}{n}\epsilon^2)$ -time [17], and since $D = O(d^2)$ we get:

Theorem 1 *The Löwner maximal matrix \bar{S} of a set of n d -dimensional symmetric matrices can be approximated by a matrix $\tilde{S} \succ \bar{S}$ such that $\text{tr}(\tilde{S}) \leq (1 + \epsilon)\text{tr}(\bar{S})$ in $O(\frac{d^2}{n}\epsilon^2)$ -time.*

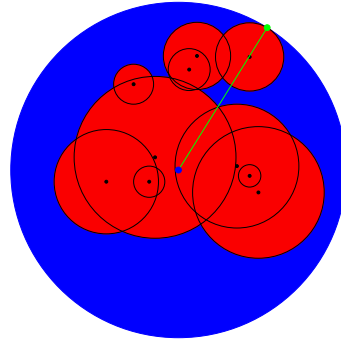
Interestingly, this shows that the approximation of Löwner supremum matrices admits core-sets [17], the subset of farthest balls $B_{f(i)}$ chosen during the l iterations, so that $\tilde{S} = \max(S_{f(1)}, \dots, S_{f(l)})$ with $\text{tr}(\tilde{S}) \leq (1 + \epsilon)\text{tr}(\bar{S})$. See [18] for other MEB approximation algorithms.

⁷<https://www.youtube.com/watch?v=w1ULgGAK6vc>

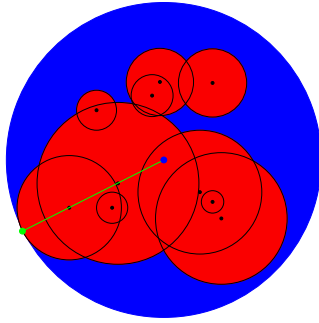
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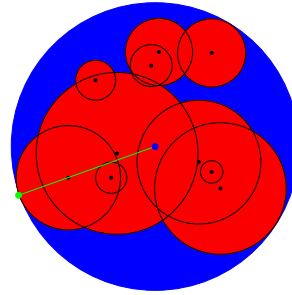
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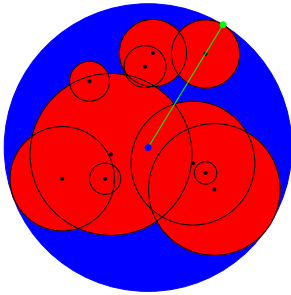
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#iteration=1008 radius=0.8658882118248943



#iteration=2008 radius=0.865753627961044



#iteration=3008 radius=0.8655840510827957

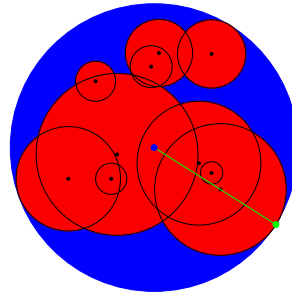


Figure 2: Approximating the minimum enclosing ball of balls iteratively: Snapshots at iterations 1, 2, 3, 1008, 2008 and 3008 (best viewed in color).

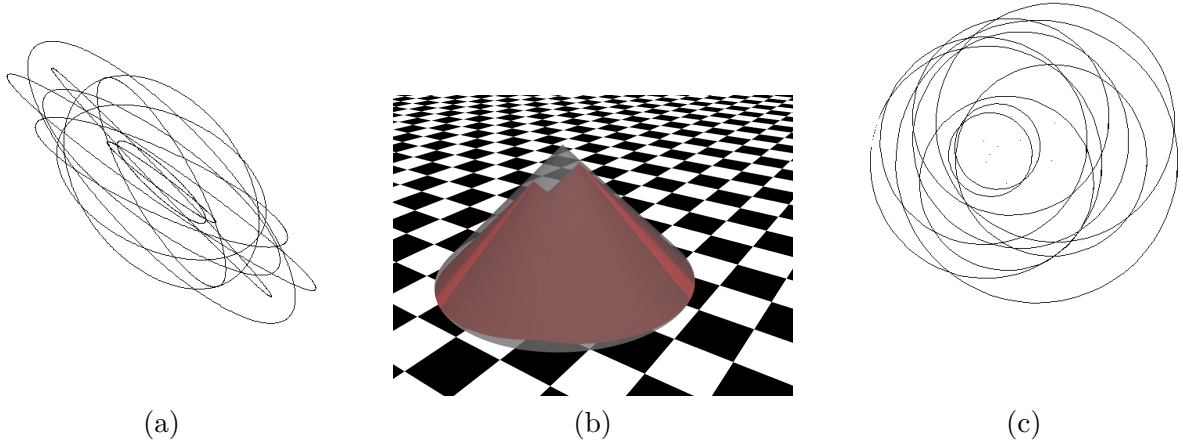


Figure 3: Equivalent visualizations: (a) 2×2 PSD matrices visualized as ellipsoids, with (b) corresponding 3D vector Löwner cones, and (c) corresponding cone vector ball basis.

To a symmetric matrix S , we associate a *quadratic form* $q_S(x) = x^\top S x$ that is a strictly convex function when S is PSD. Therefore, we may visualize the SPSD matrices in 2D/3D as ellipsoids (potentially degenerated flat ellipsoids for rank-deficient matrices). More precisely, we associate to each positive definite matrix S , a geometric ellipsoid defined by $\mathcal{E}(S) = \{x \in \mathbb{R}^d : x^\top S^{-1} x = \rho\}$, where ρ is a prescribed constant (usually set to $\rho = 1$, Figure 3). From the SVD decomposition of S^{-1} , we recover the rotation matrix, and the semi-radii of the ellipsoid are the square root eigenvalues $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d}$. It follows that $P \succeq Q \Leftrightarrow \mathcal{E}(P) \supseteq \mathcal{E}(Q)$. To handle degenerate flat ellipsoids that are not fully dimensional (rank-deficient matrix P), we define $\mathcal{E}(P) = \{x \in \mathbb{R}^d : x x^\top \preceq P\}$. Note that those ellipsoids are all centered at the origin, and may also conceptually be thought as centered Gaussian distributions (or covariance matrices denoting the concentration ellipsoids of estimators [2] in statistics). We can also visualize the Löwner ordering cone and dominance cones for 2×2 matrices embedded in the vectorized 3D space of symmetric matrices (Figure 3), and the corresponding half-vectorized ball basis (Figure 3).

4 Concluding remarks

Our novel extremal matrix approximation method allows one to leverage further related results related to core-sets [16] for dealing with high-dimensional extremal matrices. For example, we may consider clustering PSD matrices with respect to Löwner order and use the k -center clustering technique with guaranteed approximation [19, 20]. A JavaTM code of our method is available for reproducible research.

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